On decay and blow-up for a system of viscoelastic wave equations with logarithmic nonlinearity

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Abstract

In this contribution, we focus on the interplay between viscoelastic Kirchhoff term and logarithmic source term. We deal with the finite time blow-up of solutions with subcritical initial energy for a system of viscoelastic Kirchhoff type wave equations with weak damping terms and logarithmic nonlinearities. Also, the polynomial decay result was obtained. These results fill in the gaps in previous studies on this type of models.

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1 Introduction

In this paper, we deal with the following for a system of viscoelastic wave equations of Kirchhoff type with logarithmic nonlinearities and damping terms for $(x,t)\in\Omega\times(0,\infty)$

$$\begin{cases} u_{tt}-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\triangle u+\int\limits_0^t g_1(t-s)\triangle u(s)\ ds+u_t=|u|^{p-2}u\ln|u|, \\ v_{tt}-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\triangle v+\int\limits_0^t g_2(t-s)\triangle v(s)\ ds+v_t=|v|^{p-2}v\ln|v|, \end{cases} \tag{1.1}$$

with the inital-boundary conditions

$$\begin{cases} u(x,0)=u_0(x), & u_t(x,0)=u_1(x), & x\in\Omega, \\ v(x,0)=v_0(x), & v_t(x,0)=v_1(x), & x\in\Omega, \end{cases}$$

and

$$u(x,t)=0,\ v(x,t)=0,\quad (x,t)\in\partial\Omega\times[0,\infty),$$

where $p\geq 2\gamma+2$ is real number. $\Omega\subset R^n$ $(n\geq 1)$ is a regular and bounded domain with smooth boundary $\partial\Omega$. Here, M is a positive C^1 function for $s\geq 0$ satisfying $M(s)=1+s^{\gamma}$. The kernel $g_i(.):R^+\to R^+$ $(i=1,2)$ satisfies some conditions to be specified later.

Before going further, Eq. (1.1) without the logarithmic source term, Eq. (1.1) becomes the system of nonlinear Kirchhoff type equations with viscoelastic term which has been extensively studied and several results concerning existence and nonexistence have been established [8, 15, 16, 17, 19, 20, 27]. The single equation case of (1.1)

$$u_{tt}-M\left(\|\nabla u\|^2\right)\triangle u+\int\limits_0^t g(t-s)\triangle u(s)\,ds+h(u_t)=f(u), \tag{1.2}$$

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is a model to describe the motion of deformable solids as hereditary effect is incorporated [25].

In the case of $M(s)=1$, $g=0$, $f(u)=u\ln|u|$, (1.2) becomes a wave equation with logarithmic source term

$$u_{tt}-\triangle u+h(u_t)=u\ln|u|. \tag{1.3}$$

In field of mathematical physics, hyperbolic equations with this type of nonlinearities which have been appear naturally in supersymmetric field theory and inflation cosmology are one of the most important nonlinear evolution equations. Furthermore, there are applications in many branches of physics such as nuclear physics, optics and geophysics [5, 6, 11]. Because of that we should also point out that there is extensive literature on the question of existence, asymptotic behavior and nonexistence of solution for the Eq.(1.2), see in this regard [2, 3, 4, 7, 9, 10, 13, 18, 19, 21, 22, 23, 24, 28, 29, 30] .

Studying the interaction effect between logarithmic nonlinearity and other structure associated with wave system of Kirchhoff type also started very widely. Here, we mention only some results about the system of Kirchhff type hyperbolic equations with logartihmic source term. In [26], authors investigated the problem (1.1) in the case of $g=0$. They proved global existence and finite time blow up. Later, in [14], the same author of this article investigated the higher-order following system

$$\begin{cases} u_{tt}+M\left(\|D^{r_1}u\|^2+\|D^{r_2}v\|^2\right)(-\triangle)^{r_1}u-\Delta u_t+|u_t|^{q-2}u_t=|u|^{p-2}u\ln|u| \\ v_{tt}+M\left(\|D^{r_1}u\|^2+\|D^{r_2}v\|^2\right)(-\triangle)^{r_2}v-\Delta v_t+|v_t|^{q-2}v_t=|v|^{p-2}v\ln|v|. \end{cases} \tag{1.4}$$

They studied global existence and decay results of the problem (1.4). In [12], the problem (1.1) was studied with strong damping term by authors. The work focused on the interplay between strong damping term and logarithmic source term. Global existence result was established by using the potential well method with Faedo–Galerkin's method, and exponential decay rate estimates were obtained.

Motivated by above researchs, we investigated in the present work system (1.1) with nonzero $g_i(.)$ and nonconstant $M(s)$. This problem has been studied previously with the assumptions that there are no weak damping term and logarithmic source term. In this work, we note that the method ([31]) no longer applies in this particular situation when the logarithmic nonlinear term appears, for blow-up in finite time of the solutions. By providing a completely different method from previous studies, we show that the solutions may blow up at subcritical initial energy ($E(0)<d$) when the model involves the weak damping term and logarithnic source term may have different signs. Then, Theorem 4.2 shows us that there is quite a difference in cases of the equations with logarithmic nonlinear term for blow-up in finite time of the solutions. After that, the decay esitimates of solutions in the stable set were also obtained.

This paper is organized as follows. In section 2, we give some lemmas and assumptions which will be used. Later, in section 3 we state the the lemmas which used for potential well and global existence Theorem 3.5. The blow up results are presented in section 4. In section 5, we established polynomial decay result.

2 Preliminaries

We adopt the usual notations and convention for the proof of our results. Throughout this paper, for brevity of notations, we denote by $\|.\|_p$ the Lebesgue space $L^p(\Omega)$ norm and $\|.\|$ denotes $L^2(\Omega)$

norm. As usual, $(u,v) = \int\limits_{\Omega} u(x)\, v(x)\, dx$ introduce the inner product in $L^2(\Omega)$ and the duality pairing between H^{-1} and $H_0^1(\Omega)$, respectively. Particulary, C_i $(i = 1, 2, ...)$ denotes various positive constants which depend on the known constants and may be different at each appearance. First, we state general hypotheses on g :

$(A1)$ $g_i(t) : [0,\infty) \to [0,\infty)$, $(i = 1, 2)$ is a nonincreasing and differentiable function which satisfies

$$g_i(0) > 0,\ l_i = 1 - \int\limits_0^{\infty} g_i(s)\, ds,\ l = \min\{l_1, l_2\}.$$

$(A2)$ There exists positive constant ϱ such that

$$g_i'(t) \le -\varrho g_i(t),\ t \ge 0$$

and for all $t \ge 0$

$$\int\limits_0^{\infty} g_i(s)\, ds < 1 - l.$$

In order to state our results, we define the potential energy functional of problem (1.1)

$$\begin{aligned} J(u,v) &= \frac{1}{2}\left(1 - \int\limits_0^t g_1(s)\, ds\right)\|\nabla u\|^2 + \frac{1}{2}\left(1 - \int\limits_0^t g_2(s)\, ds\right)\|\nabla v\|^2 \\ &+ \frac{1}{2(\gamma+1)}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} + \frac{1}{2}\left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] \\ &- \frac{1}{p}\left(\int\limits_{\Omega} |u|^p \ln|u|\, dx + \int\limits_{\Omega} |v|^p \ln|v|\, dx\right) + \frac{1}{p^2}\left(\|u\|_p^p + \|v\|_p^p\right), \end{aligned} \tag{2.1}$$

and the Nehari functional

$$\begin{aligned} I(u,v) &= \left(1 - \int\limits_0^t g_1(s)\, ds\right)\|\nabla u\|^2 + \left(1 - \int\limits_0^t g_2(s)\, ds\right)\|\nabla v\|^2 \\ &+ \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \\ &- \left(\int\limits_{\Omega} |u|^p \ln|u|\, dx + \int\limits_{\Omega} |v|^p \ln|v|\, dx\right), \end{aligned} \tag{2.2}$$

for $u \in H_0^1(\Omega)$, where

$$(g \circ \nabla u)(t) = \int\limits_0^t g(t-s)\, \|\nabla u(s) - \nabla u(t)\|^2\, ds.$$

From the definitions (2.1) and (2.2), we can easily verify that

$$\begin{aligned} J(u,v) &= \frac{I(u,v)}{p} + \left(1-\int_0^t g_1(s)\,ds\right)\frac{(p-2)}{2p}\|\nabla u\|^2 \\ &+ \left(1-\int_0^t g_2(s)\,ds\right)\frac{(p-2)}{2p}\|\nabla v\|^2 \\ &+ \frac{(p-2\gamma-2)}{2(\gamma+1)}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} \\ &+ \frac{(p-2)}{2p}\left((g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right) \\ &+ \frac{1}{p^2}\left(\|u\|_p^p+\|v\|_p^p\right). \end{aligned} \tag{2.3}$$

Then we can introduce the stable set

$$W=\left\{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega): J(u,v)>d,\ I(u,v)>0\ \right\}\cup\{0\},$$

the outer space of the potential well

$$V=\left\{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega): J(u,v)>d, I(u,v)<0\ \right\}.$$

We denote the total energy functional associated with problem (1.1) is given by

$$\begin{aligned} E(u,v) &= \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right) \\ &+\frac{1}{2}\left(1-\int_0^t g_1(s)\,ds\right)\|\nabla u\|^2+\frac{1}{2}\left(1-\int_0^t g_2(s)\,ds\right)\|\nabla v\|^2 \\ &+\frac{1}{2(\gamma+1)}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1}+\frac{1}{2}\left[(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right] \\ &-\frac{1}{p}\left(\int_\Omega |u|^p\ln|u|\,dx+\int_\Omega |v|^p\ln|v|\,dx\right)+\frac{1}{p^2}\left(\|u\|_p^p+\|v\|_p^p\right). \end{aligned} \tag{2.4}$$

The initial energy function is obtained as

$$\begin{aligned} E(0) &= \frac{1}{2}\left(\|u_1\|^2+\|v_1\|^2\right) \\ &+\frac{1}{2}\left(1-\int_0^t g_1(s)\,ds\right)\|\nabla u_0\|^2+\left(1-\int_0^t g_2(s)\,ds\right)\|\nabla v_0\|^2 \\ &+\frac{1}{2(\gamma+1)}\left(\|\nabla u_0\|^2+\|\nabla v_0\|^2\right)^{\gamma+1}+\frac{1}{2}\left[(g_1\circ\nabla u_0)(t)+(g_2\circ\nabla v_0)(t)\right] \\ &-\frac{1}{p}\left(\int_\Omega |u_0|^p\ln|u_0|\,dx+\int_\Omega |v_0|^p\ln|v_0|\,dx\right)+\frac{1}{p^2}\left(\|u_0\|_p^p+\|v_0\|_p^p\right), \end{aligned} \tag{2.5}$$

for $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$, $t\geq 0$. Then, by (2.4) and (2.3), it is obvious that

$$E(t)=E(u,v)=\frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+J(u,v). \tag{2.6}$$

Lemma 2.1. (Sobolev-Poincare inequality [1])Let m be a number with $2\leq m<\infty$ if $n\leq 2r$ and $2\leq m\leq \frac{2n}{n-2r}$ if $n>2r$. Then there is a best constant depending on such that

$$\|u\|_m\leq C\|\nabla u\|^2,\ \forall u\in H_0^1(\Omega)$$

Lemma 2.2. $E(t)$ is a nonincreasing function for $t\geq 0$ and

$$\begin{aligned} E'(t) &= \frac{1}{2}\left[(g_1'\circ\nabla u)(t)+(g_2'\circ\nabla v)(t)\right]-\left(\|u_t\|^2+\|v_t\|^2\right) \\ &-\frac{1}{2}\left[g_1(t)\|\nabla u\|^2+g_2(t)\|\nabla v\|^2\right] \\ &\leq \frac{1}{2}\left[(g_1'\circ\nabla u)(t)+(g_2'\circ\nabla v)(t)\right] \\ &\leq 0. \end{aligned} \tag{2.7}$$

where

$$(g'\circ\nabla u)(t)=\int_0^t g'(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2\,dxdt. \tag{2.8}$$

Proof. Multiplying the first equation of (1.1) by u_t and the second equation of (1.1) by v_t, and integrating on Ω, we obtain

$$\begin{aligned} E(0) &= E(t)+\int_0^t \|u_\tau\|^2\,d\tau+\int_0^t \|v_\tau\|^2\,d\tau \\ &+\frac{1}{2}\int_0^t \left[(g_1'\circ\nabla u)(\tau)+(g_2'\circ\nabla v)(\tau)\right]d\tau \\ &-\frac{1}{2}\int_0^t \left[g_1(t)\|\nabla u\|^2+g_2(t)\|\nabla v\|^2\right]d\tau. \end{aligned} \tag{2.9}$$

for $t \geq 0$. Being the primitive of an integrable function, $E(t)$ is absolutely continuous, and by the assumptions $(A1)$, $(A2)$, (2.7) is valid. Q.E.D.

3 Potential well and existence

Later we establish some properties of the $J(u,v)$ and $I(u,v)$ respectively. We list below without proof.

Lemma 3.1. [12]. Assume that $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $\|\nabla u\| \neq 0$ and $\|\nabla v\| \neq 0$, then we get

i) $\lim\limits_{\lambda \to 0^+} J(\lambda u, \lambda v) = 0$, $\lim\limits_{\lambda \to \infty} J(\lambda u, \lambda v) = -\infty$,

ii) For $0 < \lambda < \infty$ there exists a unique λ^* such that $\frac{d}{d\lambda} J(\lambda u, \lambda v) \mid_{\lambda=\lambda^*} = 0$,

iii) $J(\lambda u, \lambda v)$ is strictly decreasing on $\lambda^* < \lambda < \infty$, $J(\lambda u, \lambda v)$ is strictly increasing on $0 \leq \lambda \leq \lambda^*$, and takes the maximum at $\lambda = \lambda^*$. In other words, there is a unique $\lambda^* \in (0,\infty)$ such that

$$I(\lambda u, \lambda v) = \begin{cases} > 0, & 0 \leq \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda. \end{cases}$$

Lemma 3.2. [12] i) The definition the depth of potential well

$$d = \inf_{u \in N} J(u,v) \tag{3.1}$$

where

$$N = \left\{(u,v) : (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{0\} : I(u,v) = 0\right\}$$

is equivalent to

$$d = \inf \left\{ \sup_{\lambda \geq 0} J(\lambda u, \lambda v) : (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega), \|\nabla u\|^2 \neq 0, \|\nabla v\|^2 \neq 0 \right\}. \tag{3.2}$$

ii) The constant d in (3.1) satisfies

$$d = \frac{l(p-2)}{2p} \left(\frac{l}{C_1^{p+1}} \right)^{\frac{2}{p-1}},$$

where C_1 is the optimal constant of Lemma 2.1 ($H_0^1(\Omega) \hookrightarrow L^{p+1}$) and

$$\begin{cases} 2\gamma + 2 \leq p \leq \frac{n+2}{n-2}, \ n > 3, \\ 2\gamma + 2 \leq p \leq \infty, \quad n = 1,2. \end{cases} \tag{3.3}$$

Lemma 3.3. [12]. Let the the conditions $(A1)$ and $(A2)$ hold. Then, for $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $(u_1, v_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$, there exist (u,v) a weak solution of problem (1.1). Suppose that $E(0) < d$.

i) If $(u_0, v_0) \in W$, then $(u,v) \in W$ for $0 \leq t \leq T$;

ii) If $(u_0, v_0) \in V$, then $(u,v) \in V$ for $0 \leq t \leq T$,

where T is the maximum existence time of $(u(t), v(t))$.

Lemma 3.4. Under the conditions of Lemma 3.3 in (i) , we obtain

$$E(0) \geq E(u,v) \geq J(u,v) > \frac{l(p-2)}{2p}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right).$$

Proof. By $I(u,v) > 0$, (2.3) and $(A1)$, we get

$$J(u,v) \geq \frac{l(p-2)}{2p}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right),$$

where $p > 2\gamma + 2$.Then by the equality of energy (2.9) and (2.6) we can see clearly that

$$E(0) \geq E(u,v) \geq J(u,v) > \frac{l(p-2)}{2p}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right).$$

Q.E.D.

Now, we are ready to state the global existence of problem (1.1) whose proof can be found in [12].

Theorem 3.5. Let $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $(u_1, v_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $(A1)$-$(A2)$ conditions hold. If $E(0) < d$ and $I(u_0, v_0) > 0$ or $\|\nabla u_0\|^2 + \|\nabla v_0\|^2 = 0$, then (1.1) admits a global weak solution $(u(t), v(t)) \in L^\infty\left(0,\infty;\ H_0^1(\Omega) \times H_0^1(\Omega)\right)$, $(u_t(t), v_t(t)) \in L^\infty\left(0,\infty;\ H_0^1(\Omega) \times\ H_0^1(\Omega)\right)$.

4 Blow up

Lemma 4.1. [31].Suppose that $\varphi(t)$ is a twice continuously differentiable function satisfying for $t > 0$

$$\begin{cases} \varphi''(t) + \varphi'(t) \geq C_0 \varphi^{1+\varsigma}(t) \\ \varphi(0) > 0 \text{ and } \varphi'(0) \geq 0 \end{cases}$$

where C_0 and ς are positive constants. Therefore $\varphi(t)$ blows up in finite time.

Theorem 4.2. Let $(A1)$, $(A2)$ and $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $(u_1, v_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ hold and (u, v) is a solution of (1.1). Assume that initial conditions satisfies $E(0) \leq 0$ and

$$\int_\Omega (u_0 u_1 + v_0 v_1) \geq 0$$

and

$$\max\left\{\int_0^t g_1(s)\,ds, \int_0^t g_2(s)\,ds\right\} \leq 1 - \frac{4p+1}{(p+1)^2}$$

then the corresponding solution blows up in finite time.

Proof. We set the following auxiliary function

$$\varphi(t) = \frac{1}{2}\int_\Omega \left(u^2 + v^2\right) dx. \tag{4.1}$$

A straightforward calculation gives

$$\varphi'(t)=\int_{\Omega}(uu_t+vv_t)\,dx. \tag{4.2}$$

Consequently from (4.2) and direct using (1.1), we have

$$\begin{aligned}
\varphi''(t) &= \|u_t\|^2+\|v_t\|^2+\int_{\Omega}(uu_{tt}+vv_{tt})\,dx\\
&= \|u_t\|^2+\|v_t\|^2-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\|\nabla u\|^2\\
&\quad -M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\|\nabla v\|^2\\
&\quad +\int_{\Omega}\int_0^t g_1(t-s)\,\nabla u(t)\,\nabla u(s)\,dsdx\\
&\quad +\int_{\Omega}\int_0^t g_2(t-s)\,\nabla u(t)\,\nabla u(s)\,dsdx\\
&\quad -\int_{\Omega}u(t)\,u_t(s)\,dx-\int_{\Omega}v(t)\,v_t(s)\,dx\\
&\quad +\int_{\Omega}u^p\ln u dx+\int_{\Omega}v^p\ln v dx.
\end{aligned} \tag{4.3}$$

Now we use Young's inequality to estimates third and fourth terms in right hands of the term (4.3); we find

$$\begin{aligned}
&\int_{\Omega}\int_0^t g_1(t-s)\,\nabla u(t)\,\nabla u(s)\,dsdx\\
\leq\ & \delta\|\nabla u\|^2+\frac{1}{4\delta}\left(\int_0^t g_1(s)\,ds\right)(g_1\circ\nabla u)(t)+\left(\int_0^t g_1(s)\,ds\right)\|\nabla u\|^2
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
&\int_{\Omega}\int_0^t g_2(t-s)\,\nabla v(t)\,\nabla v(s)\,dsdx\\
\leq\ & \delta\|\nabla v\|^2+\frac{1}{4\delta}\left(\int_0^t g_2(s)\,ds\right)(g_2\circ\nabla v)(t)+\left(\int_0^t g_2(s)\,ds\right)\|\nabla v\|^2.
\end{aligned} \tag{4.5}$$

By combining (4.3)-(4.5) to obtain

$$\begin{aligned}\varphi''(t)+\varphi'(t) \geq\ & \|u_t\|^2+\|v_t\|^2-(1+\delta+k)\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ & -\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} \\ & -\frac{k}{4\delta}\left[(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right] \\ & +\int_\Omega u^p\ln u dx+\int_\Omega v^p\ln v dx. \end{aligned} \tag{4.6}$$

where $k=\max\left\{\int_0^t g_1(s)\,ds, \int_0^t g_2(s)\,ds\right\}$.

Using the definition of $E(t)$, (4.6) becomes that

$$\begin{aligned}\varphi''(t)+\varphi'(t) \geq\ & \left(1+\frac{p}{2}\right)\|u_t\|^2+\|v_t\|^2 \\ & +\left(\frac{pl}{2}-k-1-\delta\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ & -\left(1-\frac{p}{2(\gamma+1)}\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} \\ & +\left(\frac{p}{2}-\frac{k}{4\delta}\right)\left[(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right] \\ & +\frac{1}{p}\left(\|u\|_p^p+\|v\|_p^p\right)-2pE(t). \end{aligned} \tag{4.7}$$

At this point from assumption of the theorem, $A1$, $A2$ and $\delta>\frac{k}{2p}$, we guarantee

$$\frac{p}{2}-\frac{k}{4\delta}\geq 0 \text{ and } \frac{pl}{2}-k-1-\delta\geq 0.$$

Therefore, by using of the Lemma 2.2 and $E(0)<0$ (4.7) becomes that

$$\varphi''(t)+\varphi'(t)\geq\frac{1}{p}\left(\|u\|_p^p+\|v\|_p^p\right). \tag{4.8}$$

Now, Hölder inequality are used to estimates $\|u\|_p^p$ and $\|v\|_p^p$ as follows

$$\int_\Omega |u|^2\,dx\leq\left(\int_\Omega |u|^p\,dx\right)^{\frac{2}{p}}\left(\int_\Omega 1dx\right)^{\frac{-p-2}{p}}$$

K_n is called the volume of the domain K, then

$$\|u\|_p^p\geq\left(\int_\Omega |u|^2\,dx\right)^{\frac{p}{2}}(K_n)^{\frac{2-p}{2}} \tag{4.9}$$

and by similar calculations we obtain

$$\|v\|_p^p \geq \left(\int_\Omega |v|^2 \, dx \right)^{\frac{p}{2}} (K_n)^{\frac{2-p}{2}} . \tag{4.10}$$

By inserting (4.9) and (4.10) into the (4.8), we find

$$\varphi''(t) + \varphi'(t) \geq \frac{1}{p} (K_n)^{\frac{2-p}{2}} \left[\left(\int_\Omega |u|^2 \, dx \right)^{\frac{p}{2}} + \left(\int_\Omega |v|^2 \, dx \right)^{\frac{p}{2}} \right]. \tag{4.11}$$

To estimate the right-hand side in (4.11), we make use of the following inequality

$$(a+b)^{\upsilon} \leq 2^{\upsilon - 1} (a^{\upsilon} + b^{\upsilon})$$

where $a, b \geq 0$, $1 \leq \upsilon < \infty$. Therefore, we find

$$2^{\frac{p-2}{2}} \left(\int_\Omega |u|^2 \, dx + \int_\Omega |v|^2 \, dx \right)^{\frac{p}{2}} \leq \left(\int_\Omega |u|^2 \, dx \right)^{\frac{p}{2}} + \left(\int_\Omega |v|^2 \, dx \right)^{\frac{p}{2}} .$$

Consequently, (4.11) becomes

$$\varphi''(t) + \varphi'(t) \geq 2^{p-1} \frac{1}{p} (K_n)^{\frac{2-p}{2}} (\varphi(t))^{1 + \frac{p-2}{2}}$$

It is easy to verify that the requirements of Lemma 4.1 are satisfied by

$$2^{p-1} \frac{1}{p} (K_n)^{\frac{2-p}{2}} > 0 \text{ and } \frac{p-2}{2} > 0.$$

Therefore $\varphi(t)$ blows up in finite. The proof is completed. Q.E.D.

5 Polynomial decay

Theorem 5.1. Let $(u_0, v_0) \in W$, $(u_1, v_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $A1$, $A2$ and $E(0) < d$ hold. Suppose that there is a positive fixed S_0 such that $E(t)$ satisfies the following polynomial decay estimate for $\forall t \in [0, \infty)$

$$E(u(t), v(t)) \leq \frac{S_0}{1+t}.$$

Proof. It follows from Lemma 3.3 that $u \in W$ on $[0, T]$. By using the definition of the d, $A1$, (2.9), (2.6) and (2.3) we obtain the following inequality

$$\begin{aligned} d &> E(0) \geq E(u(t), v(t)) + \int_0^t \|u_t\|^2 d\tau + \int_0^t \|v_t\|^2 d\tau \\ &= \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + J(u,v) + \int_0^t \|u_t\|^2 d\tau + \int_0^t \|v_t\|^2 d\tau \\ &\geq \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{I(u,v)}{p} + \frac{l(p-2)}{2p}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &\quad + \frac{(p-2\gamma-2)}{2(\gamma+1)}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} + \frac{1}{p^2}\left(\|u\|_p^p + \|v\|_p^p\right) \\ &\quad + \frac{(p-2)}{2p}\left((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right) + \int_0^t \|u_t\|^2 d\tau + \int_0^t \|v_t\|^2 d\tau. \end{aligned} \tag{5.1}$$

which means that

$$\frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) \leq d \tag{5.2}$$

$$\frac{2pd}{l(p-2)} \leq \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \text{ or } \frac{2(\gamma+1)d}{(p-2\gamma-2)} \leq \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1}, \tag{5.3}$$

$$\frac{2pd}{(p-2)} \leq \left((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right) \tag{5.4}$$

$$p^2 d \leq \|u\|_p^p + \|v\|_p^p \tag{5.5}$$

and

$$\int_0^t \|u_t\|^2 d\tau \leq d \text{ and } \int_0^t \|v_t\|^2 d\tau \leq d. \tag{5.6}$$

From $I(u,v) \geq 0$ and Lemma 3.1, we claim that there is a constant $\lambda^* \geq 1$ such that

$I(\lambda^* u, \lambda^* v) = 0$. Therefore, from (2.3) and (3.1), for $p \geq 2\gamma + 2$ we conclude

$$\begin{aligned}
d &\leq J(\lambda^* u, \lambda^* v) = \frac{1}{p} I(\lambda^* u, \lambda_1 v) + \frac{l(p-2)}{2p}\left(\|\nabla(\lambda^* u)\|^2 + \|\nabla(\lambda^* v)\|^2\right) \\
&\quad + \frac{(p-2\gamma-2)}{2(\gamma+1)}\left(\|\nabla\lambda^* u\|^2 + \|\nabla\lambda^* v\|^2\right)^{\gamma+1} \\
&\quad + \frac{(p-2)}{2p}\left((g_1 \circ \nabla(\lambda^* u))(t) + (g_2 \circ \nabla(\lambda^* v))(t)\right) \\
&\quad + \frac{1}{p^2}\left(\|\lambda^* u\|_p^p + \|\lambda^* v\|_p^p\right) \\
&= \frac{l(p-2)}{2p}\left(\|\nabla(\lambda^* u)\|^2 + \|\nabla(\lambda^* v)\|^2\right) \\
&\quad + \frac{(p-2\gamma-2)}{2(\gamma+1)}\left(\|\nabla(\lambda^* u)\|^2 + \|\nabla(\lambda^* v)\|^2\right)^{\gamma+1} \\
&\quad + \frac{(p-2)}{2p}\left((g_1 \circ \nabla(\lambda^* u))(t) + (g_2 \circ \nabla(\lambda^* v))(t)\right) \\
&\quad + \frac{1}{p^2}\left(\|\lambda^* u\|_p^p + \|\lambda^* v\|_p^p\right) \\
&= (\lambda^*)^{2(\gamma+1)}\left[\frac{l(p-2)}{(\lambda^*)^{2\gamma} 2p}\left(\|\nabla(u)\|^2 + \|\nabla(v)\|^2\right)\right. \\
&\quad + \frac{(p-2\gamma-2)}{2(\gamma+1)}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&\quad + \frac{(p-2)}{(\lambda^*)^{2\gamma} 2p}\left((g_1 \circ \nabla(u))(t) + (g_2 \circ \nabla(v))(t)\right) \\
&\quad \left. + \frac{1}{(\lambda^*)^{2(\gamma+1)-p} p^2}\left(\|u\|_p^p + \|v\|_p^p\right)\right] \\
&\leq (\lambda^*)^{2(\gamma+1)} J(u,v) \leq (\lambda^*)^{2(\gamma+1)} E(u,v) \\
&< (\lambda^*)^{2(\gamma+1)} E(0),
\end{aligned} \tag{5.7}$$

which satisfies that

$$\lambda^* > \left(\frac{d}{E(0)}\right)^{\frac{1}{2(\gamma+1)}} > 1. \tag{5.8}$$

On the other hand, by using $I(\lambda^* u, \lambda^* v) = 0$ equality and definition of $I(u,v)$, we get

$$\begin{aligned}
0 &= I(\lambda^* u) = (\lambda^*)^2 l \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\
&+ (\lambda^*)^{2(\gamma+1)} \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&+ (\lambda^*)^2 \left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] \\
&- \left(\int_\Omega |\lambda^* u|^p \ln |\lambda^* u| \, dx + \int_\Omega |\lambda^* v|^p \ln |\lambda^* v| \, dx\right) \\
&= (\lambda^*)^2 l \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\
&+ (\lambda^*)^{2(\gamma+1)} \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&+ (\lambda^*)^2 \left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] \\
&- (\lambda^*)^p \left(\int_\Omega |u|^p \ln |u| \, dx + \int_\Omega |v|^p \ln |v| \, dx\right) \\
&- (\lambda^*)^p \ln |\lambda^*| \left(\|u\|_p^p + \|v\|_p^p\right) \\
&= (\lambda^*)^p I(u,v) - \left[(\lambda^*)^p - (\lambda^*)^2\right] l \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\
&- \left[(\lambda^*)^p - (\lambda^*)^{2(\gamma+1)}\right] \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&- \left[(\lambda^*)^p - (\lambda^*)^2\right] \left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] \\
&- (\lambda^*)^p \ln |\lambda^*| \left(\|u\|_p^p + \|v\|_p^p\right). \qquad (5.9)
\end{aligned}$$

By combining (5.8) with (5.9), we arrive at

$$\begin{aligned} I(u) &\geq \left[\frac{(\lambda^*)^p-(\lambda^*)^2}{(\lambda^*)^p}\right]\left[l\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right] \\ &\quad \left[\frac{(\lambda^*)^p-(\lambda^*)^{2(\gamma+1)}}{(\lambda^*)^p}\right]\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1}+\ln|\lambda^*|\left(\|u\|_p^p+\|v\|_p^p\right) \\ &\geq \left[1-(\lambda^*)^{2-p}\right]\left[l\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right] \\ &\quad \left[1-(\lambda^*)^{2(\gamma+1)-p}\right]\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} \\ &= \beta_1\left[l\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right] \\ &\quad +\beta_2\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} \\ &\geq \beta\left[\begin{array}{l} l\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} \\ +(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\end{array}\right], \end{aligned} \tag{5.10}$$

where $\beta=\min\{\beta_1,\beta_2\}\in(0,1)$.

Next, we multiple the first equation of (1.1) by u and integrate over $\Omega\times(0,t)$ and the second equation of (1.1) by v and integrate over $\Omega\times(0,t)$. Then, we obtain

$$\begin{aligned} &\int_0^t\int_\Omega M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)|\nabla u|^2\,dxd\tau-\int_0^t\int_\Omega |u|^{p-1}u\ln|u|\,dxd\tau \\ =\ & -\int_0^t\int_\Omega u_{tt}udxdt-\int_0^t\int_\Omega u_tudxd\tau+\int_0^t\int_\Omega\int_0^t g_1(t-s)\nabla u(t)\nabla u(s)\ dxd\tau ds, \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} &\int_0^t\int_\Omega M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)|\nabla v|^2\,dxd\tau-\int_0^t\int_\Omega |v|^{p-1}v\ln|v|\,dxd\tau \\ =\ & -\int_0^t\int_\Omega v_{tt}vdxd\tau-\int_0^t\int_\Omega v_tvdxd\tau+\int_0^t\int_\Omega\int_0^t g_2(t-s)\nabla v(t)\nabla v(s)\ dxd\tau ds \end{aligned} \tag{5.12}$$

Since

$$\begin{aligned} &\int_\Omega\int_0^t g_1(t-s)\nabla u(t)\nabla u(s)\ dxd\tau ds \\ =\ & \frac{1}{2}\left[\int_0^t g_1(t-s)\|\nabla u(t)\|^2+\int_0^t g_1(t-s)\|\nabla u(s)\|^2-(g_1\circ\nabla u)(t)\right] \end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega}\int_{0}^{t} g_2(t-s)\nabla v(t)\nabla v(s)\ dxd\tau ds \\
= & \frac{1}{2}\left[\int_0^t g_2(t-s)\|\nabla v(t)\|^2+\int_0^t g_2(t-s)\|\nabla v(s)\|^2-(g_2\circ\nabla v)(t)\right]
\end{aligned}$$

and using the definition of $I(u,v)$ we conclude that

$$\int_0^t I(u,v)\,d\tau \le -\int_0^t\int_\Omega u_{tt}udxdt-\int_0^t\int_\Omega u_t udxd\tau-\int_0^t\int_\Omega v_{tt}vdxdt-\int_0^t\int_\Omega v_t vdxd\tau.$$

From the definition of $I(u,v)$, (5.11), (5.12) and using of Young and Hölder inequality, (5.11) implies that

$$\begin{aligned}
\int_0^t I(u,v)\,d\tau \le\ & -\int_0^t\int_\Omega u_{tt}udxdt-\int_0^t\int_\Omega u_t udxd\tau-\int_0^t\int_\Omega v_{tt}vdxdt-\int_0^t\int_\Omega v_t vdxd\tau \\
=\ & -\int_0^t\int_\Omega \frac{d}{dt}(u_t,u)\,dxdt+\int_0^t\|u_t\|^2 d\tau-\frac{1}{2}\int_0^t\frac{d}{dt}\|u\|^2 d\tau \\
& -\int_0^t\int_\Omega \frac{d}{dt}(v_t,v)\,dxdt+\int_0^t\|v_t\|^2 d\tau-\frac{1}{2}\int_0^t\frac{d}{dt}\|v\|^2 d\tau \\
=\ & \int_0^t\|u_t\|^2 d\tau-(u_t(t),u(t))+(u_1,u_0)-\frac{1}{2}\|u\|^2+\frac{1}{2}\|u_0\|^2 \\
& \int_0^t\|v_t\|^2 d\tau-(v_t(t),v(t))+(v_1,v_0)-\frac{1}{2}\|v\|^2+\frac{1}{2}\|v_0\|^2 \\
\le\ & \int_0^t\|u_t\|^2 d\tau+\int_0^t\|v_t\|^2 d\tau+\frac{1}{2}\|u\|^2+\frac{1}{2}\|v\|^2+\frac{1}{2}\|u_t\|^2 \\
& +\frac{1}{2}\|v_t\|^2+\frac{1}{2}\|u_0\|^2+\frac{1}{2}\|v_0\|^2+\frac{1}{2}\|u_1\|^2+\frac{1}{2}\|v_1\|^2 \\
& +\frac{1}{2}\|u\|^2+\frac{1}{2}\|v\|^2+\frac{1}{2}\|u_0\|^2+\frac{1}{2}\|v_0\|^2
\end{aligned} \tag{5.13}$$

Inserting (5.2) and (5.6) into (5.13), for $0<t<\infty$ we get

$$\int_0^t I(u)\,d\tau \le C. \tag{5.14}$$

Morever, the combination of (5.10) and (5.14), it follows that

$$\int_0^t l\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\le\frac{1}{1-(\lambda^*)^{2-p}}\int_0^t I(u,v)\,d\tau\le C, \tag{5.15}$$

$$\int_0^t\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1}d\tau\le\frac{1}{1-(\lambda^*)^{2(\gamma+1)-p}}\int_0^t I(u,v)\,d\tau\le C, \tag{5.16}$$

and

$$\int_0^t\left(\|u\|_p^p+\|v\|_p^p\right)d\tau\le\frac{1}{\ln|\lambda^*|}\int_0^t I(u,v)\,d\tau\le C. \tag{5.17}$$

By using Lemma 3.1, we consider that

$$\begin{aligned}[(1+t)E(t)]' &= (1+t)E'(t)+E(t)\\ &\le E(t)\end{aligned} \tag{5.18}$$

Integrating the (5.18) over $(0,t)$ and using (2.6) and (2.3), it implies that

$$\begin{aligned}(1+t)E(t) &\le E(0)+\int_0^t E(\tau)\,d\tau\\
&= E(0)+\frac{1}{2}\left(\int_0^t\|u_t\|^2d\tau+\int_0^t\|v_t\|^2d\tau\right)+\frac{1}{p}\int_0^t I(u,v)\,d\tau\\
&\quad \frac{l(p-2)}{2p}\int_0^t\left(\|\nabla u\|^2+\|\nabla v\|^2\right)d\tau\\
&\quad +\frac{(p-2)}{2p}\int_0^t((g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t))\,d\tau\\
&\quad +\frac{(p-2\gamma-2)}{2(\gamma+1)}\int_0^t\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1}d\tau\\
&\quad +\frac{1}{p^2}\int_0^t\left(\|u\|_p^p+\|v\|_p^p\right)d\tau.\end{aligned} \tag{5.19}$$

Consequently, inserting (5.6), (5.14)-(5.17) into (5.19), we prove that there is a positive fixed S_0 such that

$$(1+t)E(t)\le S_0.$$

Therefore, the proof was completed. Q.E.D.

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